Journal of Computational and Applied Mathematics 37 (1991) 221–235
North-Holland

Time-dependent scattering and transmission functions

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Received 7 September 1990

Abstract

Karanjai, S. and G. Biswas, Time-dependent scattering and transmission functions, Journal of Computational and Applied Mathematics 37 (1991) 221–235.

A system of integro-differential equations for the scattering and transmission functions for a homogeneous finite nonemitting atmosphere which scatters anisotropically is derived. A principle of invariance for the time-dependent transfer of radiation is then used to formulate the functional equations for the scattering and transmission functions.

Keywords: Transport problems, time-dependent, scattering and transmission functions.

1. Introduction

In the theory of radiative transfer in a plane-parallel atmosphere, the principles of invariance governing the laws of diffuse reflection and transmission play an important role. Following the idea of Ambartsumian, Chandrasekhar [4] stated the complete set of principles of invariance. Ueno tried to extend the principles to an inhomogeneous atmosphere of semi-infinite [19] or of finite [20] optical thickness and to a time-dependent radiation field [21]. Matsumoto [9] obtained the functional equations for the scattering and transmission functions for a finite inhomogeneous atmosphere. Matsumoto [10,11] also formulated the time-dependent principle of invariance. Bellman et al. [2] and Fymat and Ueno [6] formulated the principles for a radiation field with a finite order of scattering. Matsumoto [12] then developed the time-dependent principles of invariance for a finite order of scattering. Mullikin [17] verified rigorously the principles of invariance by virtue of functional analysis. Matsumoto [13] also obtained the functional equations for the scattering and transmission functions for a plane-parallel inhomogeneous atmosphere scattering anisotropically.

Recently Biswas and Karanjai [3,8] obtained the functional equations for the scattering and transmission functions for (i) a homogeneous isotropically scattering finite atmosphere and (ii) an anisotropically scattering finite inhomogeneous atmosphere by principles of invariance.

The theory of diffuse reflection and transmission of radiation by a finite atmosphere bounded by reflecting surface was developed for the study of planetary atmospheres. The two types of reflecting surfaces, viz. Lambert's law reflector and specular reflector, are extensively studied. Mukai [14] used the doubling method developed in [7,23] to consider the problem of reflection and transmission by planetary atmosphere with two types of reflecting surfaces. The reflection and transmission function for a finite inhomogeneous anisotropically scattering atmosphere bounded by a reflecting surface have been obtained in [22] with the help of the integral operator method. Mukai and Ueno [16] reduced Chandrasekhar's planetary problem [4] in the specular reflection case to the standard problem by using an integral equation method. Planetary problems for inhomogeneous anisotropically scattering atmospheres have been considered in [1,5,18,22]. Mukai [15] derived two systems of integro-differential equations for scattering and transmission functions of an inhomogeneous, anisotropically scattering atmosphere bounded by a reflecting surface with the aid of the principle of invariance, taking into account the polarity of the optical properties of the medium.

Here we have derived a system of integro-differential equations for scattering and transmission functions for a homogeneous finite, nonemitting atmosphere. The principle of invariance for the time-dependent transfer of radiation is then used to formulate the functional equations for the scattering and transmission functions.

2. Formulation of the problem

We consider a homogeneous, plane-parallel, nonemitting and anisotropically scattering atmosphere of finite geometrical thickness, whose optical properties vary with geometrical depth x, $x_0 \leqslant x \leqslant x_1$. The intensity of radiation $I(x, \mu, \phi, t)$ at time t, at geometrical depth x, in direction (μ, ϕ), satisfies the equation of transfer

$$\left(\frac{1}{C}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x} + 1\right) I(x, \mu, \phi, t) = J(x, t), \tag{1}$$

where C is the velocity of light.

We denote the duration of temporal capture, i.e., the mean time between successive interaction, by t_1 and assume that probability of emission during the interval of time $(t, t + \mathrm{d}t)$ is given by [10]

$$f(t)\,\mathrm{d}t = t_1 \exp\left(-\frac{t}{t_1}\right)\mathrm{d}t. \tag{2}$$

We consider scattering as absorption with a subsequent emission of a photon, and the source function $J(x, t)$ takes the form

$$J(x, t) = \frac{1}{4\pi}\int_0^t f(t - t')\,\mathrm{d}t' \times \int_{-1}^{+1}\int_0^{2\pi} J(x, \mu, \phi', t') P(x, \mu, \phi; \mu', \phi')\,\mathrm{d}\mu'\,\mathrm{d}\phi'. \tag{3}$$

The time-independent phase function $P(x, \mu, \phi; \mu', \phi')$ satisfies the following relations:

$$P(x, \mu, \phi; \mu_0, \phi_0) = P(x, \mu_0, \phi_0; \mu, \phi), \tag{4}$$

$$P(x, \mu, \phi; -\mu_0, \phi_0) = P(x, -\mu, \phi; \mu_0, \phi_0). \tag{5}$$

We have the normalization condition

$$\frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} P(x, \mu, \phi; \mu', \phi')\, \mathrm{d}\mu'\, \mathrm{d}\phi' = 1. \tag{6}$$

The initial and boundary conditions are as follows:

$$I(x, \mu, \phi, 0) = 0, \quad \text{for } t < 0, \tag{7}$$

$$I^+(x_0, \mu, \phi, t) = F\delta(\mu - \mu_0)\delta(\phi - \phi_0)\delta(t), \tag{8}$$

$$\begin{aligned} &I^-(x, \mu, \phi, t) \\ &\quad = \frac{1}{\mu}\int_0^t\int_0^1\int_0^{2\pi} I^+(x_1, \mu, \phi', t')K(\mu, \phi, \mu', \phi', t - t')\, \mathrm{d}\mu'\, \mathrm{d}\phi'\, \mathrm{d}t', \\ &\quad t > 0, \quad 0 \leqslant \mu, \mu_0 \leqslant 1, \quad 0 \leqslant \phi, \phi_0 \leqslant 2\pi, \end{aligned} \tag{9}$$

where δ is the Dirac δ-function as usual. The + and − direction mean the direction towards the lower surface $x = x_1$ and the direction towards the upper surface $x = x_0$, respectively. The boundary conditions reveal the fact that the upper surface is illuminated by a Dirac δ-function type incident radiation, and the bottom surface is a reflecting surface so that no radiation can escape from or enter into the surface. We have introduced the function $K(\mu\ \phi, \mu', \phi', t - t')$ to measure the probability that a photon in direction (μ, ϕ) and at time t will reappear after reflection in direction (μ', ϕ') at time t' from the bottom surface.

3. Functional equations for S and T functions

We divide the radiation field into two parts, the first one being the reduced incident radiation which is incident on the boundary surface and penetrates to the level x without any scattering processes. The second one is the diffuse radiation field which arises in consequence of one or more scattering processes [4]. Hence we can write

$$\begin{aligned} &I^+(x, \mu, \phi, t) \\ &\quad = I_\mathrm{d}^+(x, \mu, \phi, t) + \pi F\delta(\mu - \mu_0)\delta(\phi - \phi_0)\delta\left(t - \frac{x - x_0}{\mu_0}\right)\exp\left(-\frac{x - x_0}{\mu_0}\right), \end{aligned} \tag{10}$$

$$\begin{aligned} &I^-(x, \mu, \phi, t) \\ &\quad = I_\mathrm{d}^-(x, \mu, \phi, t) + \pi FK\left(\mu, \phi, \mu_0, \phi_0, t - \frac{x_1 - x_0}{C\mu_0}\right)\exp\left(-\frac{x_1 - x_0}{\mu_0}\right) \\ &\quad \times\exp\left(-\frac{x_1 - x}{\mu}\right)\frac{\mu_0}{\mu}, \end{aligned} \tag{11}$$

where $I_\mathrm{d}^+(x, \mu, \phi, t)$ represents the diffuse radiation field. The second term in (11) represents radiation which falls on $x = x_0$, reaches the bottom surface with attenuation and suffers reflection there and again attenuates to reach the surface $x = x_1$.

We now decompose the equation of transfer for the two directions to get

$$\left(\frac{1}{C}\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}+1\right)I^{+}(x,\mu,\phi,t)=J(x,t), \tag{12}$$

$$\left(\frac{1}{C}\frac{\partial}{\partial t}-\mu\frac{\partial}{\partial x}+1\right)I^{-}(x,\mu,\phi,t)=J(x,t). \tag{13}$$

We substitute (10) and (11) in (12) and (13) and we get after some calculations,

$$\begin{aligned}
&\left(\frac{1}{C}\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}+1\right)I_{\mathrm{d}}^{+}(x,\mu,\phi,t)\\
&=\frac{1}{4\pi}\int_0^t\int_{-1}^{+1}\int_0^{2\pi}I_{\mathrm{d}}(x,\mu,\phi',t')f(t-t')P(x,\mu,\phi;\mu',\phi')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t'\\
&+\frac{F}{4\mu}P(x,\mu,\phi;\mu_0,\phi_0)f\left(t-\frac{x-x_0}{C\mu_0}\right)\exp\left(-\frac{x-x_0}{\mu_0}\right)\\
&+\frac{F}{4\mu}\int_0^t f(t-t')\,\mathrm{d}t'\\
&\times\int_0^1\int_0^{2\pi}P(\mu,\phi;-\mu',\phi)K\left(\mu',\phi';\mu_0,\phi_0,t'-\frac{x_1-x_0}{C\mu_0}\right)\frac{\mu_0}{\mu}\,\mathrm{d}\mu'\,\mathrm{d}\phi',
\end{aligned} \tag{14}$$

where

$$I_{\mathrm{d}}(x,\mu,\phi,t)=I_{\mathrm{d}}^{+}(x,\mu,\phi,t)+I_{\mathrm{d}}^{-}(x,\mu,\phi,t). \tag{15}$$

The boundary conditions now take the form

$$I_{\mathrm{d}}^{+}(x_1,\mu,\phi,t)=0, \tag{16}$$

$$I_{\mathrm{d}}^{-}(x_1,\mu,\phi,t)=\frac{1}{\mu}\int_0^t\int_0^1\int_0^{2\pi}K(\mu,\phi;\mu',\phi',t-t')I_{\mathrm{d}}^{+}(x,\mu,\phi',t')\,\mathrm{d}\mu'\,\mathrm{d}t'. \tag{17}$$

We now proceed to formulate the principle of invariance for this particular problem. For an atmosphere with free surface conditions we can write the total radiation field as follows:

$$\begin{aligned}
&I_{\mathrm{s}}^{+}(x,\mu,\phi,t)\\
&=I_{\mathrm{ds}}^{+}(x,\mu,\phi,t)+\pi F\delta(\mu-\mu_0)\delta(\phi-\phi_0)\delta\left(t-\frac{x-x_0}{C\mu_0}\right)\exp\left(-\frac{x-x_0}{\mu_0}\right),
\end{aligned} \tag{18}$$

$$I_{\mathrm{s}}^{-}(x,\mu,\phi,t)=I_{\mathrm{ds}}^{-}(x,\mu,\phi,t), \tag{19}$$

where $I_{\mathrm{ds}}^{\pm}$ represents the diffuse radiation field for a free surface. The principle of invariance can be obtained for the diffuse radiation field in the following form:

$$\begin{aligned}
I_{\mathrm{ds}}^{-}(x,\mu,\phi,t)&=\frac{F}{4\mu}S_0\left(x_1-x,\mu,\phi;\mu_0,\phi_0,t-\frac{x-x_0}{C\mu_0}\right)\exp\left(-\frac{x-x_0}{\mu_0}\right)\\
&+\frac{1}{4\pi\mu}\int_0^t\int_0^1\int_0^{2\pi}S_0(x_1-x,\mu,\phi;\mu',\phi',t-t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t',
\end{aligned} \tag{20}$$

and

$$\frac{F}{4\mu}T_0(x_1-x_0,\mu,\phi;\mu_0,\phi_0)$$
$$=\frac{F}{4\mu}\exp\left(-\frac{x-x_0}{\mu_0}\right)T_0\left(x_1-x,\mu,\phi;\mu_0,\phi_0,t-\frac{x-x_0}{C\mu_0}\right)$$
$$+\exp\left(-\frac{x_1-x}{\mu}\right)I_{\mathrm{ds}}^{+}\left(x,\mu,\phi,t-\frac{x_1-x}{C\mu}\right)$$
$$+\frac{1}{4\pi\mu}\int_0^t\int_0^1\int_0^{2\pi}T_0(x_1-x,\mu,\phi;\mu',\phi',t-t')I_{\mathrm{ds}}^{+}(x,\mu',\phi',t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t',\tag{21}$$

where S_0 and T_0 are scattering and transmission functions respectively for the diffuse fields for free surfaces. We have

$$\frac{F}{4\mu}S_0(x_1-x_0,\mu,\phi;\mu_0,\phi_0,t)=I_{\mathrm{ds}}^{-}(x_0,\mu,\phi,t),\tag{22}$$

$$\frac{F}{4\mu}T_0(x_1-x_0,\mu,\phi;\mu_0,\phi_0,t)=I_{\mathrm{ds}}^{+}(x_1,\mu,\phi,t).\tag{23}$$

We substitute (22) and (23) in (20) and (21) and get as in the time-independent case [15]

$$I_{\mathrm{s}}^{-}(x,\mu,\phi,t)$$
$$=\frac{1}{4\pi\mu}\int_0^t\int_0^1\int_0^{2\pi}\bar{S}_0(x_1-x_0,\mu,\phi;\mu',\phi',t-t')I_{\mathrm{ds}}^{+}(x,\mu',\phi',t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t',\tag{24}$$

$$\frac{F}{4\mu}T_0(x-x_0,\mu,\phi;\mu_0,\phi_0,t)$$
$$=\exp\left(-\frac{x_1-x}{\mu}\right)I_{\mathrm{s}}^{+}\left(x,\mu,\phi,t-\frac{x_1-x}{C\mu}\right)$$
$$-\exp\left(-\frac{x_1-x}{\mu}\right)\pi F\delta(\mu-\mu_0)\delta(\phi-\phi_0)\delta\left(t-\frac{x_1-x}{C\mu}-\frac{x-x_0}{C\mu_0}\right)\exp\left(-\frac{x-x_0}{\mu_0}\right)$$
$$+\frac{1}{4\pi\mu}\int_0^t\int_0^1\int_0^{2\pi}T_0(x_1-x,\mu,\phi;\mu',\phi',t-t')I_{\mathrm{ds}}^{+}(x,\mu',\phi',t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t'.\tag{25}$$

For $x=x_1$ we get

$$\frac{F}{4\mu}T_0(x_1-x_0;\mu,\phi;\mu_0,\phi_0,t)$$
$$=I_{\mathrm{s}}^{+}(x_1,\mu,\phi,t)-\pi F\delta(\mu-\mu_0)\delta(\phi-\phi_0)\delta\left(t-\frac{x_1-x_0}{C\mu_0}\right)\exp\left(-\frac{x_1-x_0}{\mu_0}\right).\tag{26}$$

In deriving (24) we used

$$\frac{F}{4\mu}\bar{S}_0(x_1-x_0,\mu,\phi;\mu_0,\phi_0,t)=I_{\mathrm{s}}^{-}(x_0,\mu,\phi,t),\tag{27}$$

where $\bar{S}_0$ is the scattering function for the total radiation field for a free surface. If we use

$$\frac{F}{4\pi}\bar{T}_0(x_1 - x_0,\ \mu,\ \phi;\ \mu_0,\ \phi_0,\ t)$$
$$= I_s^+(x_1,\ \mu,\ \phi,\ t) = \frac{F}{4\mu}T_0(x_1 - x_0,\ \mu,\ \phi;\ \mu_0,\ \phi_0,\ t)$$
$$+ \pi F\delta(\mu - \mu_0)\delta(\phi - \phi_0)\delta\left(t - \frac{x - x_0}{C\mu_0}\right)\exp\left(-\frac{x_1 - x_0}{\mu_0}\right), \quad (28)$$

where $\bar{T}_0$ represents the transmission function for the total radiation field, we get from (26)

$$\frac{F}{4\mu}\bar{T}_0(x_1 - x,\ \mu,\ \phi;\ \mu,\ \phi_0,\ t)$$
$$= \frac{1}{4\pi\mu}\int_0^t\int_0^1\int_0^{2\pi}\bar{T}_0(x_1 - x,\ \mu,\ \phi,\ \mu',\ \phi',\ t - t')I_s^+(x,\ \mu',\ \phi',\ t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t'. \quad (29)$$

We can write

$$\bar{S}_0(x_1 - x,\ \mu,\ \phi;\ \mu',\ \phi',\ t) = S_0(x_1 - x,\ \mu,\ \phi,\ \mu',\ \phi',\ t), \quad (30)$$
$$\bar{T}_0(x_1 - x,\ \mu,\ \phi;\ \mu',\ \phi',\ t) = T_0(x_1 - x,\ \mu,\ \phi,\ \mu',\ \phi',\ t). \quad (31)$$

Equations (24), (30) and (28), (31) form the principles of invariance for the problem for the standard atmosphere. We now consider the reflecting bottom surface which is sometimes designated as the planetary atmosphere where $K(\mu,\ \phi;\ \mu',\ \phi',\ t) \neq 0$, as in the standard atmosphere. It is apparent that the principle of invariances can be written down in the above form for the planetary atmosphere. We write

$$I^-(x,\ \mu,\ \phi,\ t)$$
$$= \frac{1}{4\pi\mu}\int_0^t\int_0^1\int_0^{2\pi}\bar{S}(x_1 - x,\ \mu,\ \phi;\ \mu',\ \phi',\ t - t')I^+(x,\ \mu',\ \phi',\ t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t', \quad (32)$$

$$\frac{F}{4\mu}\bar{T}(x_1 - x,\ \mu,\ \phi;\ \mu_0',\ \phi_0,\ t)$$
$$= \frac{1}{4\pi\mu}\int_0^t\int_0^1\int_0^{2\pi}\bar{T}(x_1 - x,\ \mu,\ \phi;\ \mu',\ \phi',\ t - t')I^+(x,\ \mu',\ \phi',\ t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t', \quad (33)$$

where we have defined the scattering $\bar{S}$ and transmission $\bar{T}$ function as the probability of ultimate reflection and transmission, respectively in direction $(\mu,\ \phi)$ at time t and direction $(-\mu_0,\ \phi_0)$ at time t', which is incident on the upper surface of the atmosphere of optical thickness $x_1 - x_0$ at time t'. These functions satisfy relations like (30) and (33).

Now we substitute (10) and (11) in (32) to get

$$I_d^-(x,\ \mu,\ \phi,\ t) + \frac{\mu_0}{\mu}FK\left(\mu,\ \phi;\ \mu_0,\ \phi_0,\ t - \frac{x_1 - x}{C\mu_0}\right)\exp\left(-\frac{x_1 - x_0}{\mu_0}\right)\exp\left(-\frac{x_1 - x}{\mu}\right)\pi$$
$$= \frac{1}{4\pi\mu}\int_0^t\int_0^1\int_0^{2\pi}\bar{S}(x_1 - x,\ \mu,\ \phi;\ \mu',\ \phi',\ t - t')I_d^+(x,\ \mu',\ \phi',\ t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t'$$
$$+ \frac{1}{4\pi\mu}\int_0^t\int_0^1\int_0^{2\pi}\bar{S}(x_1 - x,\ \mu,\ \phi;\ \mu',\ \phi',\ t - t')\pi F\delta(\mu - \mu_0)$$
$$\times\,\delta(\phi' - \phi_0)\delta\left(t' - \frac{x - x_0}{C\mu_0}\right)\exp\left(-\frac{x - x_0}{\mu'}\right)\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t'. \quad (34)$$

We now use

$$\bar{S}(x_1-x, \mu, \phi, \mu', \phi', t) = S(x_1-x, \mu, \phi; \mu', \phi') + 4\pi \exp\left(-\frac{x_1-x}{\mu}\right) \exp\left(-\frac{x_1-x}{\mu'}\right)\mu' K(\mu, \phi; \mu', \phi', t) \tag{35}$$

in (34) and we get

$$\begin{aligned} I_d^-(x, \mu, \phi, t) = {} & \frac{F}{4\mu} S\left(x_1 - x_0, \mu, \phi, \mu_0, \phi_0, t - \frac{x-x_0}{C\mu_0}\right) \exp\left(-\frac{x-x_0}{C\mu_0}\right) \\ & + \frac{1}{4\pi\mu} \int_0^t \int_0^1 \int_0^{2\pi} S(x_1-x, \mu, \phi, \mu', \phi', t-t') \\ & \qquad \times I_d^+(x, \mu', \phi', t')\, d\mu'\, d\phi'\, dt' \\ & + \exp\left(-\frac{x_1-x}{\mu}\right) \frac{F}{4\mu} \int_0^t \int_0^1 \int_0^{2\pi} K(\mu, \phi; \mu', t-t') I_d^+(x, \mu', \phi', t') \\ & \qquad \times \exp\left(-\frac{x_1-x}{\mu'}\right) d\mu'\, d\phi'\, dt', \end{aligned} \tag{36}$$

where S represents the scattering functions for a diffuse field.

We substitute (10) in (33) to get,

$$\begin{aligned} & \frac{F}{4\mu} \bar{T}(x_1-x_0, \mu, \phi; \mu_0, \phi_0, t) \\ & = \frac{1}{4\pi\mu} \int_0^t \int_0^1 \int_0^{2\pi} \bar{T}(x_1-x, \mu, \phi; \mu', \phi', t-t') \pi F \delta(\mu'-\mu_0)\delta(\phi'-\phi_0) \\ & \quad \times \left(t' - \frac{x-x_0}{C\mu_0}\right) \exp\left(-\frac{x-x_0}{\mu_0}\right) d\mu'\, d\phi'\, dt' \\ & \quad + \frac{1}{4\pi\mu} \int_0^t \int_0^1 \int_0^{2\pi} \bar{T}(x_1-x, \mu, \phi; \mu', \phi', t-t') I_d^+(x, \mu', \phi', t')\, d\mu'\, d\phi'\, dt'. \end{aligned} \tag{37}$$

We use

$$\begin{aligned} & \bar{T}(x_1-x, \mu, \phi; \mu', \phi', t-t') \\ & \quad = T(x_1-x, \mu, \phi, \mu', \phi', t-t') \\ & \qquad + 4\pi\delta(\mu-\mu')\delta(\phi-\phi')\delta\left(t - \frac{x_1-x}{C\mu'} - t'\right) \exp\left(-\frac{x_1-x}{\mu'}\right) \end{aligned} \tag{38}$$

in (37) and we get

$$\begin{aligned} & \frac{F}{4\mu} T(x_1-x_0, \mu, \phi; \mu_0, \phi_0, t) \\ & \quad + \frac{F}{4\mu}\delta(\mu-\mu_0)\delta(\phi-\phi_0)\delta\left(t - \frac{x_1-x_0}{C\mu_0}\right) \exp\left(-\frac{x_1-x_0}{\mu_0}\right)\mu_0 \end{aligned}$$

$$
\begin{aligned}
&= \frac{F}{4\mu} \exp\left(-\frac{x-x_0}{\mu_0}\right) T\left(x_1 - x, \mu, \phi; \mu_0, \phi_0, t - \frac{x-x_0}{C\mu_0}\right) \\
&\quad + \frac{F}{4\mu} \delta(\mu - \mu_0)\delta(\phi - \phi_0)\delta\left(t - \frac{x_1 - x}{C\mu_0} - \frac{x_1 - x}{C\mu_0}\right)\mu_0 \exp\left(-\frac{x-x_0}{\mu_0}\right) \\
&\quad \times \exp\left(-\frac{x_1 - x}{\mu_0}\right) \\
&\quad + \exp\left(-\frac{x_1 - x}{\mu}\right) I_d^+\left(x, \mu, \phi, t - \frac{x_1 - x}{C\mu}\right) \\
&\quad + \frac{1}{4\pi\mu} \int_0^t \int_0^1 \int_0^{2\pi} T(x_1 - x, \mu, \phi; \mu', \phi', t - t') I_d^+(x, \mu', \phi', t')\, d\mu'\, d\phi'\, dt'.
\end{aligned} \tag{39}
$$

From this we get,

$$
\begin{aligned}
&\frac{F}{4\mu} T(x_1 - x_0, \mu, \phi; \mu_0, \phi_0, t) \\
&= \frac{F}{4\mu} T\left(x_1 - x, \mu, \phi; \mu_0, \phi_0, t - \frac{x - x_0}{C\mu_0}\right) \exp\left(-\frac{x - x_0}{\mu_0}\right) \\
&\quad + \exp\left(-\frac{x_1 - x}{\mu}\right) I_d^+\left(x, \mu, \phi, t - \frac{x_1 - x}{C\mu}\right) \\
&\quad + \frac{1}{4\pi\mu} \int_0^t \int_0^1 \int_0^{2\pi} T(x_1 - x, \mu, \phi; \mu', \phi', t - t') I_d^+(x, \mu', \phi', t')\, d\mu'\, d\phi'\, dt'.
\end{aligned} \tag{40}
$$

Equations (36) and (40) together with equations analogous to (30) and (31) for the planetary problem constitute the mathematical description of the time-dependent principle of invariance for the planetary atmosphere with the reflecting bottom surface and incidence on the upper surface.

The required equation satisfied by the scattering and transmission function will now be derived following the general method of [4,10].

We differentiate (36) with respect to x and take the limit $x \to x_0$, and make some rearrangement of terms to get

$$
\begin{aligned}
&\lim_{x \to x_0} \frac{d}{dx} I_d^-(x, \mu, \phi, t) \\
&= \frac{F}{4\mu}\left[\frac{1}{\mu_0} + \frac{1}{C\mu_0}\frac{\partial}{\partial t} - \frac{\partial}{\partial x_0}\right] S(x_1 - x_0, \mu, \phi; \mu_0, \phi_0, t) \\
&\quad + \frac{1}{4\pi\mu} \int_0^t \int_0^1 \int_0^{2\pi} S(x_1 - x_0, \mu, \phi; \mu', \phi', t - t') \\
&\qquad\qquad \times \left[\frac{d}{dx} I_d^+(x, \mu', \phi', t')\right]_{x = x_0} d\mu'\, d\phi'\, dt'
\end{aligned}
$$

$$+\frac{\mathrm{d}}{\mathrm{d}x}\left[\exp\left(-\frac{x_1-x}{\mu}\right)\frac{F}{4\mu}\int_0^t\int_0^1\int_0^{2\pi}K(\mu,\phi;\mu',\phi',t-t')I_\mathrm{d}^+(x,\mu,\phi',t')\right.$$
$$\left.\times\exp\left(-\frac{x_1-x}{\mu'}\right)\mu'\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t'\right]_{x=x_0}. \tag{41}$$

We have from (14)

$$\lim_{x\to x_0}\frac{\mathrm{d}}{\mathrm{d}x}I_\mathrm{d}^+(x,\mu,\phi',t')=\frac{J(t',x_0)}{\mu'}, \tag{42}$$

where

$$J(t',x_0)=\frac{1}{4\pi}\int_0^{t'}f(t'-t'')\,\mathrm{d}t''$$
$$\times\int_0^1\int_0^{2\pi}I_\mathrm{d}^+(x_0,\mu',\phi',t'')P(x_0,\mu',\phi';\mu'',\phi'')\,\mathrm{d}\mu''\,\mathrm{d}\phi''$$
$$+\frac{1}{4\pi}\int_0^{t'}f(t'-t'')\,\mathrm{d}t''$$
$$\times\int_0^1\int_0^{2\pi}I_\mathrm{d}^-(x_0,\mu'',\phi'',t'')P(\mu',\phi';-\mu'',\phi'')\,\mathrm{d}\mu''\,\mathrm{d}\phi''$$
$$+\frac{1}{4\pi}\int_0^{t'}f(t'-t'')\,\mathrm{d}t''$$
$$\times\int_0^1\int_0^{2\pi}P(x_0,\mu',\phi';\mu'',\phi'')\pi F\delta(\mu''-\mu_0)\delta(\phi''-\phi_0)$$
$$\times\delta\left(t''-\frac{x_1-x_0}{C\mu_0}\right)\exp\left(-\frac{x_1-x_0}{\mu_0}\right)\mathrm{d}\mu''\,\mathrm{d}\phi''$$
$$+\frac{1}{4\pi}\int_0^{t}f(t'-t'')\,\mathrm{d}t''$$
$$\times\int_0^1\int_0^{2\pi}P(x_0,\mu',\phi';-\mu'',\phi'')\pi F$$
$$\times K\left(\mu'',\phi'';\mu_0,\phi_0,t''-\frac{x_1-x_0}{C\mu_0}\right)\exp\left(-\frac{x_1-x_0}{\mu_0}\right)$$
$$\exp\left(-\frac{x_1-x_0}{\mu_0}\right)\exp\left(-\frac{x_1-x_0}{\mu''}\right)\mathrm{d}\mu''\,\mathrm{d}\phi''. \tag{43}$$

We use (16) and (17) to get

$$J(t',x_0)=\frac{1}{4\pi}\int_0^{t'}f(t'-t'')\,\mathrm{d}t''\int_0^1\int_0^{2\pi}S(x_1-x_0,\mu'',\phi'';\mu_0,\phi_0,t'')$$
$$\times\frac{F}{4\mu''}P(x_0,\mu',\phi';-\mu'',\phi'')\,\mathrm{d}\mu''\,\mathrm{d}\phi''$$

$$+\frac{F}{4\mu}P(x_0, \mu', \phi'; \mu_0, \phi_0)f(t')$$
$$+\frac{F}{4\mu}\int_0^{t'} f(t'-t'')\,\mathrm{d}t''$$
$$\times\int_0^1\int_0^{2\pi} P(x_0, \mu', \phi'; -\mu'', \phi'')K\left(\mu'', \phi''; \mu_0, \phi_0, t''-\frac{x_1-x_0}{C\mu_0}\right)\frac{\mu_0}{\mu''}$$
$$\times\exp\left(-\frac{x_1-x_0}{\mu_0}\right)\exp\left(-\frac{x_1-x_0}{\mu''}\right)\mathrm{d}\mu''\,\mathrm{d}\phi''. \qquad (44)$$

We also have, from (14),

$$\frac{\mathrm{d}}{\mathrm{d}x}I_\mathrm{d}^-(x, \mu, \phi, t) = -\frac{J(t, x)}{\mu} + \frac{1}{\mu}\left(\frac{1}{C}\frac{\partial}{\partial t}+1\right)I_\mathrm{d}^-(x, \mu, \phi, t).$$

This becomes, on using (22),

$$\frac{\mathrm{d}}{\mathrm{d}x}I_\mathrm{d}^-(x, \mu, \phi, t) = -\frac{J(t, x)}{\mu} + \frac{1}{\mu}\left(\frac{1}{C}\frac{\partial}{\partial t}+1\right)\frac{F}{4\mu}S(x_1-x, \mu, \phi; \mu_0, \phi_0, t). \qquad (45)$$

We now substitute (44) and (45) in (41) and get after some lengthy rearrangement of terms,

$$-\frac{F}{4\mu}\frac{\partial}{\partial x_0}S(x_1-x_0, \mu, \phi; \mu_0, \phi_0, t)$$
$$+\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\left(\frac{1}{C}\frac{\partial}{\partial t}+1\right)\frac{F}{4\mu}S(x_1-x_0, \mu, \phi; \mu_0, \phi_0, t)$$
$$=\frac{F}{4\mu}P(x_0, \mu, \phi; \mu_0, \phi_0)f(t)$$
$$+\frac{F}{4\mu}\frac{1}{4\pi}\int_0^t f(t-t')\,\mathrm{d}t'\int_0^1\int_0^{2\pi} S(x_1-x_0, \mu', \phi; \mu_0, \phi_0, t')$$
$$\times P(x_0, \mu, \phi; -\mu', \phi')\frac{\mathrm{d}\mu'}{\mu'}\mathrm{d}\phi$$
$$+\frac{F}{4\mu}\int_0^t f(t-t')\,\mathrm{d}t\int_0^1\int_0^{2\pi} P(x_0, \mu, \theta; -\mu', \phi')K\left(\mu', \phi'; \mu_0, \phi_0, t'-\frac{x_1-x_0}{C\mu_0}\right)$$
$$\times\exp\left(-\frac{x_1-x_0}{\mu_0}\right)\exp\left(-\frac{x_1-x_0}{\mu'}\right)\mathrm{d}\mu'\,\mathrm{d}\phi'$$
$$+\frac{F}{4\mu}\frac{1}{16\pi^2}\int_0^t\int_0^{2\pi}\int_0^1 S(x_1-x_0, \mu, \phi; \mu', \phi', t-t')$$
$$\times\int_0^{t'}\int_0^{2\pi}\int_0^1 S(x_1-x_0, \mu'', \phi'', \mu_0, \phi_0, t'')$$
$$\times P(x, \mu', \phi'; -\mu'', \phi'')$$
$$f(t'-t'')\frac{\mathrm{d}\mu''}{\mu''}\frac{\mathrm{d}\mu'}{\mu'}\mathrm{d}\phi''\,\mathrm{d}\phi'\,\mathrm{d}t''\,\mathrm{d}t'$$

$$
\begin{aligned}
&+\frac{F}{4\mu}\frac{1}{4\pi}\int_0^t\int_0^{2\pi}\int_0^1 S(x_1-x_0,\mu,\phi;\mu',\phi',t-t')P(x_0,\mu',\phi';\mu_0,\phi_0)\\
&\qquad\qquad\times f(t')\frac{\mathrm{d}\mu'}{\mu'}\mathrm{d}\phi'\,\mathrm{d}t'\\
&+\frac{F}{4\mu}\frac{1}{4\pi}\int_0^t\int_0^{2\pi}\int_0^1 S(x_1-x_0,\mu,\phi;\mu',\phi',t-t')\int_0^{t'}f(t'-t'')\,\mathrm{d}t''\\
&\qquad\qquad\times\int_0^{2\pi}\int_0^1 P(x_0,\mu,\phi;-\mu',\phi';-\mu'',\phi'')\\
&\qquad\qquad\times K\left(\mu'',\phi'';\mu_0,\phi_0,t''-\frac{x_1-x_0}{C\mu_0}\right)\\
&\qquad\qquad\times\exp\left(-\frac{x_1-x_0}{\mu_0}\right)\mu_0\\
&\qquad\qquad\times\exp\left(-\frac{x_1-x_0}{\mu''}\right)\frac{\mathrm{d}\mu''}{\mu''}\frac{\mathrm{d}\mu'}{\mu'}\mathrm{d}\phi''\,\mathrm{d}\phi'\,\mathrm{d}t'\\
&+\frac{F}{4\mu}\exp\left(-\frac{x_1-x_0}{\mu}\right)\frac{F}{16\pi}\\
&\times\int_0^t\int_0^{2\pi}\int_0^1 K(\mu,\phi;\mu',\phi',t-t')\exp\left(-\frac{x_1-x_0}{\mu'}\right)\int_0^{t'}f(t'-t'')\,\mathrm{d}t''\\
&\qquad\int_0^{2\pi}\int_0^1 S(x_1-x_0,\mu'',\phi'',\mu_0,\phi_0,t'')\\
&\qquad\times P(x_0,\mu',\phi';-\mu'',\phi'')\frac{\mathrm{d}\mu''}{\mu''}\frac{\mathrm{d}\mu'}{\mu'}\mathrm{d}\phi''\,\mathrm{d}\phi'\,\mathrm{d}t'\\
&+\frac{F}{4\mu}\exp\left(-\frac{x_1-x_0}{\mu}\right)\frac{F}{16}\int_0^t\int_0^{2\pi}\int_0^1 K(\mu,\phi;\mu',\phi',t-t')\exp\left(-\frac{x_1-x_0}{\mu'}\right)\\
&\qquad\qquad\qquad\times P(x_0,\mu',\phi';\mu_0,\phi_0)f(t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t'\\
&+\frac{F}{4}\frac{F}{4\mu}\exp\left(-\frac{x_1-x_0}{\mu}\right)\\
&\times\int_0^t\int_0^{2\pi}\int_0^1 K(\mu,\phi;\mu',\phi';t-t')\exp\left(-\frac{x_1-x_0}{\mu'}\right)\int_0^{t'}f(t'-t'')\,\mathrm{d}t''\\
&\times\int_0^{2\pi}\int_0^1 P(x_0,\mu',\phi';-\mu'',\phi'')\\
&\qquad\times K\left(\mu'',\phi'';\mu_0,\phi_0,t''-\frac{x_1-x_0}{C\mu_0}\right)\exp\left(-\frac{x_1-x_0}{\mu_0}\right)\\
&\qquad\times\exp\left(-\frac{x_1-x_0}{\mu''}\right)\frac{\mathrm{d}\mu''}{\mu'}\mathrm{d}\phi''\,\mathrm{d}\phi'\,\mathrm{d}t'.
\end{aligned}
\tag{46}
$$

Equation (46), thus derived, is a complicated equation whose solution can be obtained under suitable boundary conditions.

We now differentiate (40) with respect to x and take the limit as $x \to x_0$ to get,

$$\begin{aligned} &-\frac{F}{4\mu}\frac{\partial}{\partial x_0}T(x_1-x_0,\mu,\phi;\mu_0,\phi_0,t) \\ &+\left(\frac{1}{C}\frac{\partial}{\partial t}+1\right)\frac{1}{\mu_0}\frac{F}{4\mu}T(x_1-x_0,\mu,\phi;\mu_0,\phi_0,t) \\ &=\frac{\mathrm{d}}{\mathrm{d}x}\left[\exp\left(-\frac{x_1-x}{\mu}\right)I_{\mathrm{d}}^{+}\left(x,\mu,\phi,t-\frac{x_1-x}{C\mu}\right)\right]_{x=x_0} \\ &+\frac{1}{4\pi\mu}\int_0^t\int_0^{2\pi}\int_0^1 T(x_1-x_0,\mu,\phi;\mu',\phi' t-t') \\ &\qquad\times\left[\frac{\mathrm{d}}{\mathrm{d}x}I_{\mathrm{d}}^{+}(x,\mu',\phi',t')\right]_{x=x_0}\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t'. \end{aligned} \tag{47}$$

We shall use the following equation:

$$\begin{aligned} &\lim_{x\to x_0}\frac{\mathrm{d}}{\mathrm{d}x}I_{\mathrm{d}}^{+}\left(x,\mu,\phi,t-\frac{x_1-x}{C\mu}\right) \\ &=\lim_{x\to x_0}\int_{-\infty}^{\infty}\delta\left(t'-\left(t-\frac{x_1-x_0}{C\mu}\right)\right)\frac{\mathrm{d}}{\mathrm{d}x}I_{\mathrm{d}}^{+}(x,\mu,\phi,t')\,\mathrm{d}t'. \end{aligned} \tag{48}$$

We use (42) via (43) and substitute (47) in (48) to get, using the properties of the Dirac δ-function,

$$\begin{aligned} &-\frac{F}{4\mu}\frac{\partial}{\partial x_0}T(x_1-x_0,\mu,\phi_0,t)+\left(\frac{1}{C}\frac{\partial}{\partial t}+1\right)\frac{1}{\mu_0}T(x_1-x_0,\mu,\phi;\mu_0,t)\frac{F}{4\mu} \\ &=\exp\left(-\frac{x_1-x_0}{\mu}\right)\int_{-\infty}^{\infty}\delta\left(t'-\left(t-\frac{x_1-x}{C\mu}\right)\right)\left[\frac{\mathrm{d}}{\mathrm{d}x}I_{\mathrm{d}}^{+}(x,\mu,\phi,t')\right]_{x=x_0}\mathrm{d}t' \\ &+\frac{1}{4\pi\mu}\int_0^t\int_0^{2\pi}\int_0^1 T(x_1-x_0,\mu',\phi';\mu,\phi;t-t') \\ &\qquad\times\left[\frac{1}{4\pi}\int_0^{t'}f(t-t'')\,\mathrm{d}t''\right. \\ &\qquad\times\int_0^1\int_0^{2\pi}S(x_1-x_0,\mu'',\phi'';\mu_0,t'') \\ &\qquad\qquad\times\frac{F}{4\mu''}P(x_0,\mu'',\phi';-\mu'',\phi'')\,\mathrm{d}\mu''\,\mathrm{d}\phi'' \\ &\qquad+\tfrac{1}{4}FP(x_0,\mu',\phi';\mu_0,\phi_0)f(t'') \\ &\qquad+\tfrac{1}{4}F\int_0^{t'}f(t'-t'')\,\mathrm{d}t'' \\ &\qquad\times\int_0^1\int_0^{2\pi}P(x_0,u',\phi';-\mu'',\phi'') \\ &\qquad\qquad\times K\left(\mu'',\phi'';\mu_0,\phi_0,t''-\frac{x_1-x_0}{C\mu_0}\right) \end{aligned}$$

$$\times \frac{\mu_0}{\mu''} \exp\left(-\frac{x_1 - x_0}{\mu_0}\right) \exp\left(-\frac{x_1 - x_0}{\mu''}\right) \mathrm{d}\mu''\, \mathrm{d}\phi'' \Bigg] \mathrm{d}\mu'\, \mathrm{d}\phi'\, \mathrm{d}t'. \tag{49}$$

We get after several rearrangements,

$$\begin{aligned}
&-\frac{F}{4\mu} \frac{\partial}{\partial x_0} T(x_1 - x_0, \mu, \phi; \mu_0, \phi_0, t) \\
&+\left(\frac{1}{C} \frac{\partial}{\partial t} + 1\right) \frac{1}{\mu_0} \frac{F}{4\mu} T(x_1 - x_0, \mu, \phi; \mu_0, \phi_0, t) \\
&= \exp\left(-\frac{x_1 - x_0}{\mu}\right) \int_{-\infty}^{+\infty} \delta\left(t' - \left(t - \frac{x_1 - x_0}{C\mu}\right)\right) \frac{F}{4\mu} \\
&\qquad \times \left[\frac{1}{4\pi} \int_0^{t'} f(t' - t'')\, \mathrm{d}t''\right. \\
&\qquad\quad \times \int_0^1 \int_0^{2\pi} P(x_0, \mu, \phi; -\mu', \phi') \\
&\qquad\qquad \left.\times S(x_1 - x_0, \mu', \phi'; \mu_0, \phi_0, t'') \frac{\mathrm{d}\mu''}{\mu'} \mathrm{d}\phi'\right] \mathrm{d}t' \\
&\quad + \exp\left(-\frac{x_1 - x_0}{\mu'}\right) \frac{F}{4\mu} P(x_0, \mu, \phi; -\mu', \phi') f\left(t - \frac{x_1 - x_0}{C\mu}\right) \\
&\quad + \frac{F}{4\mu} \exp\left(-\frac{x_1 - x_0}{\mu}\right) \\
&\qquad \times \int_{-\infty}^{+\infty} \delta\left(t' - \left(t - \frac{x_1 - x_0}{C\mu}\right)\right) \\
&\qquad\quad \times \left[\int_0^{t'} f(t' - t'')\, \mathrm{d}t''\right. \\
&\qquad\quad \int_0^1 \int_0^{2\pi} P(x_0, \mu, \phi; -\mu', \phi') \\
&\qquad\qquad \times K\left(\mu', \phi'; \mu_0, \phi_0, t'' - \frac{x_1 - x_0}{C\mu_0}\right) \mu_0 \exp\left(-\frac{x_1 - x_0}{\mu_0}\right) \\
&\qquad\qquad \left.\times \exp\left(-\frac{x_1 - x_0}{\mu'}\right) \frac{\mathrm{d}\mu'}{\mu'}\, \mathrm{d}\phi'\right] \mathrm{d}t' \\
&\quad + \frac{F}{4\mu} \frac{1}{16\pi} \int_0^t \int_0^{2\pi} \int_0^1 T(x_1 - x_0, \mu', \phi'; \mu_0, \phi, t - t') \\
&\qquad\qquad \times \int_0^{t'} f(t' - t'')\, \mathrm{d}t'' \\
&\qquad\qquad \times \int_0^1 \int_0^{2\pi} S(x_1 - x_0, \mu'', \phi'', \mu_0, \phi_0, t'') \\
&\qquad\qquad\qquad \times P(x_0, \mu', \phi'; -\mu', \phi'') \frac{\mathrm{d}\mu''}{\mu''}\, \mathrm{d}\phi''\, \mathrm{d}\mu'\, \mathrm{d}\phi'\, \mathrm{d}t'
\end{aligned}$$

$$
\begin{aligned}
&+\frac{F}{4\mu}\frac{1}{4\pi}\int_0^t\int_0^{2\pi}\int_0^1 T(x_1-x_0,\mu',\phi';\mu_0,\phi_0,t-t') \\
&\qquad\times P(x_0,\mu',\phi';\mu_0,\phi_0)f(t')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t' \\
&+\frac{F}{4\mu}\frac{1}{4\pi}\int_0^t\int_0^{2\pi}\int_0^1 T(x_1-x_0,\mu',\phi';\mu_0,\phi_0,t-t') \\
&\qquad\times\int_0^{t'} f(t'-t'')\,\mathrm{d}t'' \\
&\qquad\times\int_0^1\int_0^{2\pi} P(x_0,\mu',\phi';-\mu'',\phi'') \\
&\qquad\qquad\times K\left(\mu'',\phi'';\mu_0,\phi_0,t''-\frac{x_1-x_0}{C\mu_0}\right) \\
&\qquad\qquad\times\mu_0\exp\left(-\frac{x_1-x_0}{\mu_0}\right) \\
&\qquad\qquad\times\exp\left(-\frac{x_1-x_0}{\mu''}\right)\frac{\mathrm{d}\mu''}{\mu''}\mathrm{d}\mu'\,\mathrm{d}\phi'\,\mathrm{d}t'.
\end{aligned} \tag{50}
$$

Equations (40) and (50) are the required functional equations for the scattering and transmission function for the present case. These equations are subject to initial conditions

$$S(x_1-x_0,\mu,\phi;\mu_0,\phi_0,0)=0, \tag{50a}$$

$$T(x_1-x_0,\mu,\phi;\mu_0,\phi_0,0)=0. \tag{50b}$$

It is noticed that if we let $K(\mu,\phi;\mu_0,\phi_0,t)=0$ in (46) and (50), the equations represent the corresponding equations in standard time-independent atmosphere [15].

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